Three-Loop Results in QCD with Wilson Fermions

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We calculate the third coefficient of the lattice beta function in QCD with Wilson fermions, extending the pure gauge results of Lüscher and Weisz; we show how this coefficient modifies the scaling function on the lattice. We also calculate the three-loop average plaquette in the presence of Wilson fermions. This allows us to compute the lattice scaling function both in the standard and energy schemes.

1. INTRODUCTION

In lattice formulations of field theories the relevant region for continuum physics is the one where scaling is verified. Scaling in the continuum limit requires that all RG-invariant dimensionless ratios of physical quantities approach their continuum value with non-universal corrections which get depressed by integer powers of the inverse correlation length. Tests of asymptotic scaling involve the relationship between the bare coupling and the cutoff as an essential ingredient. This relationship is expressed in the bare lattice β function which reads, in standard notation:

$$\beta^L(g_0) \equiv -a\frac{\mathrm{d}g_0}{\mathrm{d}a}\mid_{g_r,\mu} = -b_0^L g_0^3 - b_1^L g_0^5 - b_2^L g_0^7 - \ldots \quad (1)$$

The first two coefficients in Eq. (1) are scheme-independent and well known. Our purpose is to compute the third coefficient, b_2^L, in $SU(N)$ Yang-Mills theory with N_f species of Wilson fermions.

M. Lüscher and P. Weisz [1] have calculated this coefficient in the pure gluonic case, using a coordinate space method for evaluating the lattice integrals. We have verified their result [2], using a different technique where firstly the integrands are Taylor expanded in powers of the external momenta and then computed term by term with the introduction of an IR regulator. In the present work, we are extending our calculation to include contributions from Wilson fermions.

We also present here the calculation of the 3-loop coefficient w_3 of the average plaquette, in QCD with Wilson fermions. In the absence of fermions, this quantity was calculated by some of the present authors [3].

Our results for b_2^L and w_3 allow us to study the corrections to asymptotic scaling in Monte Carlo simulations of dimensionful quantities, for both the standard and energy schemes.

The involved algebra of lattice perturbation theory was carried out by making use of an extensive computer code developed by us in recent years [4]; for the purposes of the present calculation, the code was extended to include fermions and form factors.

2. CALCULATION OF β_2^L

We performed this calculation using the background field method. In order facilitate comparison we have adopted the notation of Ref.[1].

We set out to compute $d_2(\mu a)$ in:

$$\alpha_{\overline{\mathrm{MS}}}(\mu) = \alpha_0 + d_1(\mu a)\alpha_0^2 + d_2(\mu a)\alpha_0^3 + \ldots,$$

$$\alpha_0 = g_0^2/4\pi, \quad \alpha_{\overline{\mathrm{MS}}}(\mu) = g^2(\mu)/4\pi \quad (2)$$

As we shall see, b_2^L will follow immediately from

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this quantity. $d_2(\mu a)$ can be expressed as:

$$d_2(\mu a)=(4\pi)^2\{(\nu_R^{(1)}(p)-\nu^{(1)}(p))^2-\nu_R^{(2)}(p)+\nu^{(2)}(p) - \lambda(\partial\nu_R^{(1)}(p)/\partial\lambda)(\omega^{(1)}(p)-\omega_R^{(1)}(p))\}_{\lambda=\lambda_0} \quad (3)$$

in terms of the gauge parameter λ, and of $\omega(p)$ and $\nu(p)$ (the gauge- and background- two point lattice functions, respectively):

$$\sum_\mu \Gamma^{(2,0,0)}(p,-p)^{ab}_{\mu\mu}= -\delta^{ab}3\hat{p}^2[1-\nu(p)]/g_0^2$$

$$\sum_\mu \Gamma^{(0,2,0)}(p,-p)^{ab}_{\mu\mu}= -\delta^{ab}\hat{p}^2[3(1-\omega(p))+\lambda_0]$$

$$\nu(p)=\sum_{l=1}^{\infty} g_0^{2l}\nu^{(l)}(p)$$
$$\omega(p)=\sum_{l=1}^{\infty} g_0^{2l}\omega^{(l)}(p) \quad (4)$$

The subscript R refers to the analogous $\overline{\rm MS}$ renormalized quantities.

The 1-loop quantitites appearing above, $\nu^{(1)}, \nu_R^{(1)}, \omega^{(1)}, \omega_R^{(1)}$, are known. For the $\overline{\rm MS}$ 2-loop function $\nu_R^{(2)}(p)$ there are four new diagrams with respect to the pure gluonic case. We find:

$$\nu_R^{(2)}(\lambda{=}1,\,p)(16\pi^2)^2 = N^2[8\rho+577/18-6\zeta(3)]+ N_f[(-3\rho-401/36)N+(\rho+55/12-4\zeta(3))/N]$$
$$\text{with}: \quad \rho=\ln(\mu^2/p^2). \quad (5)$$

The only remaining quantity to calculate (and by far the most complicated) is $\nu^{(2)}(p)$. The inclusion of fermions brings in 20 additional diagrams (of these, two correspond to fermion mass renormalization).

The evaluation of these diagrams requires tremendous analytical effort. We have developed an extensive computer code for analytic computations in lattice perturbation theory [4]. In a nutshell, this computer code performs: Contraction among the appropriate vertices; simplification of color/Dirac matrices; use of trigonometry and momentum symmetries for reduction to a more compact, canonical form; treatment of (sub)divergences and extraction of logarithms by opportune additions/subtractions (in the case at hand, this step gives rise to hundreds of *types* of expressions, each containing typically hundreds of terms); automatic generation of highly optimized Fortran code for the loop integration of each type of expression.

The integrals are then performed numerically on finite lattices. Our programs perform extrapolations of each expression to a broad spectrum of functional forms of the type: $\sum_{i,j} e_{ij}(\ln L)^j/L^i$, analyze the quality of each extrapolation using a variety of criteria and assign statistical weights to them, and finally produce a quite reliable estimate of the systematic error.

Several consistency checks can be performed on the separate contributions of each diagram: $\mathcal{O}(p^0)$ parts obey several relations, and must sum up to zero by gauge invariance; Lorentz non-invariant terms $\sum_\mu p_\mu^4/p^2$ must cancel; the single and double logarithms are related to known results. We have performed all these checks both analytically and numerically; this was also a useful verification of the correctness of our error estimates.

Our final result for $\nu^{(2)}(p)$ is (we take $r=1$ for the Wilson parameter):

$$\nu^{(2)}(p)=\nu^{(2)}(p)|_{N_f=0} \quad (6)$$
$$+N_f[(-1/N+3N)\ln(p^2)/(16\pi^2)^2 -0.000706(3)/N+0.000544(5)*N]$$

Also, we have:

$$\nu^{(1)}(p)-\nu^{(1)}(N_f{=}0,p)=\omega^{(1)}(p)-\omega^{(1)}(N_f{=}0,p) = N_f*[\ln(p^2)/24\pi^2-0.01373219(1)] \quad (7)$$

A table of partial results will appear in a forthcoming publication [5], where we also expect to provide better accuracy.

From the relations given above, we can now easily construct $d_1(\mu a)$ and $d_2(\mu a)$. Writing: $d_1(\mu a)=d_{10}+d_{11}\ln(\mu a)$, $\quad d_2(\mu a)=(d_1(\mu a))^2+d_{20}+d_{21}\ln(\mu a)$, the coefficient b_2^L follows immediately:

$$b_2^L=b_2^{\overline{\rm MS}}+(d_{20}d_{11}-d_{21}d_{10})/(128\pi^3) \quad (8)$$

where

$$b_2^{\overline{\rm MS}} = \frac{-1}{108(4\pi)^6}[-5714N^3 + N_f^2(-224N + 66/N) + N_f(3418N^2 - 561 - 27/N^2)] \quad (9)$$

In particular:

$$b_2^L(N{=}3, N_f{=}3, r{=}1) = -0.002284(3) \quad (10)$$

$$(\text{cf.} \quad b_2^L(N{=}3, N_f{=}0) = -0.00159983) \quad (11)$$

The deviation, $q = (b_1^2 - b_2^L b_0)/(2b_0^3)$, in the asymptotic scaling formula:

$$a\Lambda_L = \exp(-1/2b_0 g_0^2)(b_0 g_0^2)^{-b_1/2b_0}\left[1 + q g_0^2 + \ldots\right]$$

is now found to be:

$$q(N{=}3, N_f{=}3, r{=}1) = 0.3694(4) \quad (12)$$

3. THE 3-LOOP MEAN PLAQUETTE

We have computed the free energy to 3 loops

$$-\ln Z/V = \text{const.} + F_2 g_0^2 + F_3 g_0^4 + \ldots \quad (13)$$

in QCD with Wilson fermions. A total of 24 diagrams involving fermions contribute to F_3.

The equality:

$$\langle 1 - \Box/N\rangle = -(1/6V)(\partial \ln Z/\partial\beta) \quad (14)$$

(valid by virtue of $\langle S_{\rm fermion}\rangle = 0$ beyond tree level) relates F_2 and F_3 directly to the quantities w_2 and w_3 in the expansion of the average plaquette:

$$\langle 1 - \Box/N\rangle = w_1 g_0^2 + w_2 g_0^4 + w_3 g_0^6 + \ldots \quad (15)$$

We report below the final values for w_2 and w_3 at $N{=}3, N_f{=}3, r{=}1$. A breakdown of the results, along with the explicit N- and N_f-dependence and improved error estimates will be presented in a forthcoming publication [6].

From the plaquette coefficients, we may now construct the coupling constant in the "energy" scheme:

$$g_E^2 = g_0^2 + (w_2/w_1) g_0^4 + (w_3/w_1) g_0^6 + \ldots \quad (16)$$

In terms of this improved definition of the coupling constant, the β function reads:

$$b_2^E = b_2^L + (b_0 w_3 - b_1 w_2 - b_0 w_2^2/w_1)/w_1 \quad (17)$$

Now, the scaling violation parameter $q = (b_1^2 - b_2^E b_0)/(2b_0^3)$ becomes (at $N = 3, N_f = 3, r = 1$):

$$q(k = 0.1675) = 0.2103(4)$$
$$q(k = 0.1560) = 0.1764(4) \quad (18)$$

This indeed represents an improvement over the standard case. We are presently working on improving the accuracy of our results, for the longer write-ups [5,6].

k	$10^3 \times w_2$	$10^3 \times w_3$
0.156	18.2522(2)	9.15(1)
0.1575	17.7713(2)	8.91(1)
0.16	16.9767(1)	8.51(1)
0.164	15.7243(2)	7.89(1)
0.1675	14.6480(6)	7.35(1)

Table 1
Values of $w_2 \times 10^3$ and $w_3 \times 10^3$. N=3, N_f=3, r=1, k: hopping parameter.

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